JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 25, No. 3, May–June 2002

H_∞ Antiwindup Design for Linear Systems Subject to Input Saturation

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It is known that actuator saturation can lead to deterioration in the performance of the control system and may even lead to instability. The invariant set and $\mathcal{L}_2$-gain problems of the linear systems subject to actuator saturation are considered. An antiwindup technique is used to enlarge the invariant set with a guaranteed prespecified $\mathcal{L}_2$ gain. The design of the antiwindup compensation gain is formulated and solved as an iterative optimization problem with linear matrix inequality constraints. Numerical examples are used to demonstrate the effectiveness of the proposed design technique.

Introduction

ONE of the common, but difficult, issues in control design is actuator saturation because all control devices are subject to saturation (limited in force, torque, current, flow rate, etc.). Those who are interested in reconfiguring the controller to compensate for unforeseen failure must also be concerned with saturation because, during reconfiguration, one is frequently forced to drive certain actuators into saturation to compensate for a loss of performance due to a failure in part of the airframe. The analysis and synthesis of controllers for dynamic systems subject to actuator saturation have been attracting increasingly more attention. Recent books [1,2] and a special journal issue[3] reported much research activity on this topic. There are mainly two approaches to dealing with actuator saturation. One approach is to take control constraints into account at the outset of the control design. Considerable progress has been made in this direction. In Ref. 1, the semiglobal stabilization problem and low-gain stabilizing approach are analyzed for linear systems subject to actuator saturation. In Ref. 4, a convex optimization method was proposed to analyze the regional stability of linear systems with input saturation. This approach was further studied by using the Popov criteria to reduce the conservativeness.[5] Disturbance attenuation and the H_∞ control problem for systems with input saturation were studied in Refs. 6–8. These results are also extended to output feedback design for systems with actuator amplitude and rate saturation in Refs. 9 and 10. In Refs. 2 and 11, a new local controller design method was proposed to enlarge the invariant set of the system with input saturation.

Another approach is to first ignore actuator saturation and design a linear controller that meets required performance specifications and then to design an antiwindup compensator to weaken the adverse influence of saturation. This method was widely used in control engineering, and there are many techniques for designing an antiwindup compensator. One of the first attempts to analyze windup control system stability was the application of the scalar Popov and circle criteria by Glattfelder and Schaufelberger[12,13] in the context of antireset windup control systems. The multivariable circle criterion was extended in Ref. 14 to design multivariable antireset windup control systems. Zheng et al.[15] used the off-axis circle criterion to establish stability of their antiwindup scheme based on the internal model control approach. Attempts to use the scaled small-gain theorem for the same purpose were reported in Ref. 16. The other methods reported are function analysis,[17] incremental gain analysis,[18] invariant subspace technique,[19] and multiplier theory.[20] A unified framework for the study of antiwindup design was presented in Ref. 21 that is known to be equivalent to the observer type approach.[22] Recently, Cao et al.[23] described the development of an antiwindup design method that leads to significant enlargement of the domain of attraction.

In this paper, we will explore the possibility of improving the performance for systems with actuator saturation by antiwindup compensation. We will present an invariant set approach to analyzing the H_∞ disturbance rejection performance of the antiwindup control systems. We will first obtain an estimate of the invariant set under a given antiwindup compensation gain. This estimate is then maximized over the choice of antiwindup compensation gain. It is known that, in the absence of disturbance, the estimates of the domain of attraction made by small-gain theorem, Popov criterion, or circle criterion are sometimes conservative.[4,5,8,24] In Ref. 11, a less conservative analysis approach was proposed to analyze the stability, domain of attraction, and invariance of linear systems with actuator saturation. The idea is to formulate the analysis problem into a constrained optimization problem with constraints given by a set of linear matrix inequalities (LMIs). In this paper, we will extend the method of Ref. 11 to arrive at an estimate of the invariant set with a guaranteed a priori specified $\mathcal{L}_2$ gain of the closed-loop system under a given antiwindup compensation gain. An iterative LMI-based approach will be proposed to design the antiwindup compensator for the enlargement of the invariant set with a given $\mathcal{L}_2$ gain.

The rest of the paper is organized as follows. The next section will give some preliminary results and state more precisely our problem formulation. This will be followed by a section where stability and the invariant set of the closed-loop system with actuator saturation and an antiwindup compensator will be analyzed. An iterative LMI algorithm will be proposed to design antiwindup compensation gain by enlarging the invariant set. Then, the $\mathcal{L}_2$ gain of the closed-loop system with antiwindup compensation will be analyzed, and the iterative algorithm will be extended to enlarge the invariant set with the given $\mathcal{L}_2$ gain by the antiwindup compensation. Finally, two numerical examples will be used to illustrate our design procedure and its effectiveness.

Throughout the paper, we will use standard notation. $\mathcal{R}$ denotes the set of real numbers, $\mathcal{R}^m$ denotes the m-dimensional Euclidean space, and $\mathcal{R}^{n\times m}$ denotes the set of all real $n\times m$ matrices. In

Received 27 April 2001; revision received 16 January 2002; accepted for publication 18 January 2002.

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the sequel, if not explicitly stated, matrices are assumed to have compatible dimensions. The notation $M > (\geq, <, \leq)0$ is used to denote a symmetric positive definite (positive semidefinite, negative definite, negative semidefinite) matrix, and $\lambda_{\min}(M)$ and $\lambda_{\max}(M)$ denote the minimum and the maximum eigenvalue of a symmetric matrix M. $\mathcal{L}_2^m([0, T))$ denotes the space of square integrable vector valued functions over $[0, T)$, that is, all functions such that

$$\|\boldsymbol{x}\|_2 = \left(\int_0^T \|\boldsymbol{x}(t)\|^2 \, \mathrm{d}t\right)^{\frac{1}{2}} < \infty$$

where we have used $\| \|$ to denote the Euclidean norm and $\| \|_2$ the $\mathcal{L}_2$ norm in $\mathcal{L}_2[0, T)$. For a vector-valued bounded function z: $[0, \infty) \to \mathcal{R}^k$, the $\mathcal{L}_\infty$ norm of z is defined as $\|z\|_\infty = \operatorname{ess\,sup}_{t\geq 0} \|z(t)\|$. Given a linear operator $T_{zw}: \mathcal{L}_\infty \to \mathcal{L}_\infty$, the induced $\mathcal{L}_\infty$ norm of T_{zw} is defined as $\|T_{zw}\|_\infty = \sup_{\|\boldsymbol{w}\|_\infty \leq 1} \|T_{zw}\boldsymbol{w}\|_\infty$.

Problems Statement and Preliminaries

Let us consider a linear system with input saturation

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B_1\boldsymbol{w}(t) + B_2\sigma[\boldsymbol{u}(t)] \tag{1}$$

$$z(t) = C_1\boldsymbol{x}(t) + D_1\boldsymbol{w}(t) \tag{2}$$

$$\boldsymbol{y}(t) = C_2\boldsymbol{x}(t) \tag{3}$$

where $\boldsymbol{x} \in \mathcal{R}^n$ is the state vector with $\boldsymbol{x}(0) = \boldsymbol{x}_0$; $\boldsymbol{u} \in \mathcal{R}^m$ the control input vector; $z \in \mathcal{R}^{p_1}$ the controlled output vector; $\boldsymbol{y} \in \mathcal{R}^{p_2}$ the measured output vector; $\boldsymbol{w} \in \mathcal{R}^q$ the exogenous disturbance input vector with $\boldsymbol{w} \in \mathcal{L}_2[0, \infty)$; and A, B_1, B_2, C_1, C_2, and D_1 are real-valued matrices of appropriate dimensions. Without loss of generality, we also assume that the bounded disturbance $\boldsymbol{w}(t)$ belongs to the set $\mathcal{W} := \{\boldsymbol{w} : \boldsymbol{w}(t)^T\boldsymbol{w}(t) \leq 1, \forall t \geq 0\}$. The function $\sigma: \mathcal{R}^m \to \mathcal{R}^m$ is the standard saturation function defined as follows:

$$\sigma(\boldsymbol{u}) = [\sigma(\boldsymbol{u}_1) \quad \sigma(\boldsymbol{u}_2) \quad \cdots \quad \sigma(\boldsymbol{u}_m)]^T$$

where $\sigma(\boldsymbol{u}_i) = \operatorname{sign}(\boldsymbol{u}_i)\min\{1, |\boldsymbol{u}_i|\}$. Here we have slightly abused the notation by using σ to denote both the scalar-valued and the vector-valued saturation functions. Also note that it is without loss of generality to assume unity saturation level. The nonunity saturation level can be absorbed into the B_2 matrix and $\boldsymbol{u}$ (see subsequent example).

We will assume that a linear dynamic compensator of the form

$$\dot{\boldsymbol{x}}_c(t) = A_c\boldsymbol{x}_c(t) + B_c\boldsymbol{y}(t), \qquad \boldsymbol{x}_c(0) = 0 \tag{4}$$

$$\boldsymbol{u}(t) = C_c\boldsymbol{x}_c(t) + D_c\boldsymbol{y}(t) \tag{5}$$

has been designed to stabilize the system (1–3) in the absence of actuator saturation and to meet a given H_∞ performance criteria. In Ref. 25, all controllers for the general H_∞ control problem are given, and a parameterization is made by using LMIs.

When the actuator saturates, the performance of the closed-loop system designed without considering the saturation may seriously deteriorate. In general, the performance degradation is caused by the states of the controller achieving different values from those in the absence of actuator saturation.[14] This can result in improper control signals and, consequently, deterioration in closed-loop performance, and may induce a limit cycle or an unstable output response. A well-known example of performance degradation, for example, large overshoot and large settling time, occurs when a linear compensator with integrators, for instance, a proportional–integral–derivative (PID) compensator, is used in a closed-loop system and the reset-windup phenomenon appears. During the time when the actuators saturate, the error is continuously integrated even though the controls are not what they should be, and, hence, the states of the compensator attain values that lead to controls larger than the saturation limit. This is known as the windup phenomenon. A common approach in practice is to perform a linear control design, then to add extra feedback compensation at the stage of control implementation, using the difference between the controller output and the actuator output signal. Because this form of compensation aims to reduce the undesirable effects of windup, it is referred to as antiwindup compensation.

A typical antiwindup compensator involves adding a correction term of the form $E_c[\sigma(\boldsymbol{u}) - \boldsymbol{u}]$. The modified compensator has the form

$$\dot{\boldsymbol{x}}_c(t) = A_c\boldsymbol{x}_c(t) + B_c\boldsymbol{y}(t) + E_c\{\sigma[\boldsymbol{u}(t)] - \boldsymbol{u}(t)\} \tag{6}$$

$$\boldsymbol{u}(t) = C_c\boldsymbol{x}_c(t) + D_c\boldsymbol{y}(t) \tag{7}$$

where $\boldsymbol{x}_c \in \mathcal{R}^{n_c}$ and $\boldsymbol{x}_c(0) = 0$. Obviously, with such a correction term, the compensator (6) and (7) would continue to behave like the dynamic controller (4) and (5) in the absence of saturation, that is, $\sigma(\boldsymbol{u}) = \boldsymbol{u}$, and the compensation in Eqs. (6) and (7) has been made in a sufficiently smooth manner so that existence and uniqueness of solutions for the closed-loop system are guaranteed.

Under the modified compensator (6) and (7), the closed-loop system can be written as

$$\dot{\tilde{\boldsymbol{x}}}(t) = \tilde{A}\tilde{\boldsymbol{x}}(t) + \tilde{B}_1\boldsymbol{w}(t) + \tilde{B}_2(\sigma(\boldsymbol{u}) - \boldsymbol{u}) \tag{8}$$

$$z(t) = \tilde{C}_1\tilde{\boldsymbol{x}}(t) + D_1\boldsymbol{w}(t) \tag{9}$$

$$\boldsymbol{u}(t) = F\tilde{\boldsymbol{x}}(t) \tag{10}$$

where

$$\tilde{\boldsymbol{x}} = \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{x}_c \end{bmatrix}, \qquad \tilde{A} = \begin{bmatrix} A + B_2D_cC_2 & B_2C_c \\ B_cC_2 & A_c \end{bmatrix}, \qquad \tilde{B}_1 = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$

$$\tilde{B}_2 = \begin{bmatrix} B_2 \\ E_c \end{bmatrix}, \qquad \tilde{C}_1 = [C_1 \quad 0], \qquad F = [D_cC_2 \quad C_c]$$

For an initial state $\tilde{\boldsymbol{x}}(0) = \tilde{\boldsymbol{x}}_0$, denote the state trajectory of the closed-loop system under $\boldsymbol{w} \in \mathcal{W}$ as $\varphi(t, \tilde{\boldsymbol{x}}_0, \boldsymbol{w})$. Our primary concern is the boundness of the trajectories of state $\varphi(t, \tilde{\boldsymbol{x}}_0, \boldsymbol{w})$ and controlled output z.

Definition 1: Consider the closed-loop system (8–10) with $\boldsymbol{w} = 0$. The set of all $\tilde{\boldsymbol{x}}_0$ such that

$$\lim_{t\to\infty} \varphi(t, \tilde{\boldsymbol{x}}_0, 0) \to 0$$

is said to be the domain of attraction of the closed-loop system equilibrium $\tilde{\boldsymbol{x}} = 0$.

Definition 2: Consider the closed-loop system (8–10). A set in $\mathcal{R}^n$ is said to be invariant if all of the trajectories $\varphi(t, \boldsymbol{x}_0, \boldsymbol{w})$ starting from within it will remain in it regardless of $\boldsymbol{w} \in \mathcal{W}$. An ellipsoid $\Omega(P, \rho) = \{\tilde{\boldsymbol{x}} : V(\tilde{\boldsymbol{x}}) = \tilde{\boldsymbol{x}}^T P\tilde{\boldsymbol{x}} \leq \rho\}$ is said to be strictly invariant if $\dot{V} = 2\tilde{\boldsymbol{x}}^T P\{\tilde{A}\tilde{\boldsymbol{x}} + \tilde{B}_2[\sigma(\boldsymbol{u}) - \boldsymbol{u})] + \tilde{B}_1\boldsymbol{w}\} < 0$ for all $\boldsymbol{w}^T\boldsymbol{w} \leq 1$ and all $\tilde{\boldsymbol{x}} \in \partial\Omega(P, \rho)$, the boundary of $\Omega(P, \rho)$.

We are interested in the following two problems.

Problem 1: Set Invariance Analysis. Let the compensation gain E_c be known.

1) For a given $P > 0$ and $\rho > 0$, determine if the ellipsoid $\Omega(P, \rho)$ is (strictly) invariant.

2) Or, determine the values of $P > 0$ and $\rho > 0$ such that the ellipsoid $\Omega(P, \rho)$ is invariant and is as large as possible.

Problem 2: Invariant Set Enlargement. Given a bounded set $X_R \subset \mathcal{R}^n$, find a compensation gain E_c, if any, such that the closed-loop system has a bounded invariant set $\Omega(P, \rho)$ that contains αX_R with α maximized. In general, we may set X_R as the invariant ellipsoid of the closed-loop system without compensation gain E_c, that is, $E_c = 0$.

Because X_R determines the general shape of the resulting $\Omega(P, \rho)$, we will refer to such an X_R as a shape reference set.

Definition 3: For a given set $X_\infty \subset \mathcal{R}^n$, the system (8–10) with initial state $\tilde{\boldsymbol{x}}(0) = 0$ is said to have a regional $\mathcal{L}_2$ gain less than or equal to γ, in X_∞, for some $\gamma > 0$ if $\boldsymbol{x}(t) \in X_\infty$ for all $t \in [0, \infty)$ and

$$\int_0^T \|z(t)\|^2 \, \mathrm{d}t \leq \gamma^2 \int_0^T \|\boldsymbol{w}(t)\|^2 \, \mathrm{d}t, \qquad \forall \boldsymbol{w} \in \mathcal{L}_2[0, T)$$

for all $T \geq 0$.

For the given system (1–3), suppose that the dynamic control law (4) and (5) is designed such that the closed-loop system is asymptotically stable and has an $\mathcal{L}_2$ gain less than or equal to γ_0 (or, for example, it has H_∞ performance index γ_0). In the presence of saturation, the global stability may be lost, and the H_∞ performance will degrade, in general. An interesting problem is how to enlarge the region of attraction with a guaranteed a priori given H_∞ performance index $\gamma \geq \gamma_0$. This problem can be formulated as follows.

Problem 3: Regional H_∞ Control Problem. For a given set $X_R \subset \mathcal{R}^n$ and a given $\gamma \geq \gamma_0$, design an antiwindup compensation gain E_c, if any, such that the following holds:

1) In the absence of disturbance, the closed-loop system is locally asymptotically stable at $\tilde{x} = 0$, and $\Omega(P, \rho)$ is contained in its domain of attraction.

2) The closed-loop system has a regional $\mathcal{L}_2$ gain less than or equal to γ in $\Omega(P, \rho) \supset \alpha X_R$, and α is maximized.

Problem 1 can be readily solved by the method proposed in Ref. 11. The objective of this paper is to provide a solution to Problems 2 and 3. In what follows, we will briefly recapitulate the method of Ref. 11 for the solution of Problem 1. Our solution to Problems 2 and 3 will be built on it.

With the feedback control law in Eq. (10), system (8) can be rewritten as

$$\dot{\tilde{x}}(t) = (\tilde{A} - \tilde{B}_2 F)\tilde{x}(t) + \tilde{B}_2\sigma(\boldsymbol{u}) + \tilde{B}_1\boldsymbol{w}(t) \tag{11}$$

Let $\boldsymbol{f}_i$ stand for the ith row of the matrix F. We define the symmetric polyhedron

$$\mathcal{L}(F) = \left\{\tilde{x} \in \mathcal{R}^{n+n_c} : |\boldsymbol{f}_i\tilde{x}| \leq 1, i = 1, \ldots, m\right\}$$

Let $P \in \mathcal{R}^{(n+n_c)\times(n+n_c)}$ be a positive-definite matrix. Define the ellipsoid

$$\Omega(P, \rho) = \{\tilde{x} \in \mathcal{R}^{n+n_c} : \tilde{x}^T P\tilde{x} \leq \rho\}$$

We would like to determine if $\Omega(P, \rho)$ is a strictly invariant set for the closed-loop system (11), or, in the absence of disturbance, if it is contained in the domain of attraction of the origin. To this end, for any two matrices F and $H \in \mathcal{R}^{m\times(n+n_c)}$ and a vector $\boldsymbol{\nu} \in \mathcal{R}^m$ denote

$$M(\boldsymbol{\nu}, F, H) := \mathrm{diag}\{\nu_1, \nu_2, \ldots, \nu_m\}F + (I - \mathrm{diag}\{\nu_1, \nu_2, \ldots, \nu_m\})H = \begin{bmatrix} \boldsymbol{\nu}_1\boldsymbol{f}_1 + (1-\boldsymbol{\nu}_1)\boldsymbol{h}_1 \\ \boldsymbol{\nu}_2\boldsymbol{f}_2 + (1-\boldsymbol{\nu}_2)\boldsymbol{h}_2 \\ \cdots \\ \boldsymbol{\nu}_m\boldsymbol{f}_m + (1-\boldsymbol{\nu}_m)\boldsymbol{h}_m \end{bmatrix}$$

where $\boldsymbol{f}_i$ and $\boldsymbol{h}_i$ denote the ith row of F and H, respectively. Let $\mathcal{V} = \{\nu \in \mathcal{R}^m : \nu_i = 1 \text{ or } 0\}$. There are 2^m elements in $\mathcal{V}$. We will use a $\boldsymbol{\nu} \in \mathcal{V}$ to choose from the rows of F and H to form a new matrix $M(\boldsymbol{\nu}, F, H)$: If $\boldsymbol{\nu}_i = 1$, then the ith row of $M(\boldsymbol{\nu}, F, H)$ is $\boldsymbol{f}_i$, and, if $\boldsymbol{\nu}_i = 0$, then the ith row of $M(\boldsymbol{\nu}, F, H)$ is $\boldsymbol{h}_i$.

We recall the following two theorems from Ref. 11.

Theorem 1: Given an ellipsoid $\Omega(P, \rho)$, if there exists an $H \in \mathcal{R}^{m\times(n+n_c)}$ such that

$$[\tilde{A} - \tilde{B}_2F + \tilde{B}_2M(\boldsymbol{\nu}, F, H)]^T P + P[\tilde{A} - \tilde{B}_2F + \tilde{B}_2M(\boldsymbol{\nu}, F, H)] < 0, \qquad \forall\boldsymbol{\nu} \in \mathcal{V} \tag{12}$$

and $\Omega(P, \rho) \subset \mathcal{L}(H)$, that is, $|\boldsymbol{h}_i x| \leq 1$ for all $\tilde{x} \in \Omega(P, \rho)$, $i = 1, 2, \ldots, m$, then $\Omega(P, \rho)$ is a contractively invariant set of system (11).

Theorem 2: For a given ellipsoid $\Omega(P, \rho)$, if there exist an H and a positive number η such that

$$\Psi(\nu) := [\tilde{A} - \tilde{B}_2F + \tilde{B}_2M(\boldsymbol{\nu}, F, H)]^T P + P[\tilde{A} - \tilde{B}_2F + \tilde{B}_2M(\boldsymbol{\nu}, F, H)] + \eta^{-1}P\tilde{B}_1\tilde{B}_1^T P + (\eta/\rho)P < 0, \qquad \forall\boldsymbol{\nu} \in \mathcal{V} \tag{13}$$

and $\Omega(P, \rho) \subset \mathcal{L}(H)$, then $\Omega(P, \rho)$ is a strictly invariant set for system (8).

It is easy to see that Problem 1 can be recast into the following optimization problem,

$$\max_{P>0, H, \rho>0, \eta>0} \quad \Omega(P, \rho) \tag{14}$$

so that

$$\Psi(\boldsymbol{\nu}) < 0, \qquad \forall\boldsymbol{\nu} \in \mathcal{V} \tag{14a}$$

$$|\boldsymbol{h}_i\tilde{x}| \leq 1, \qquad \forall\tilde{x} \in \Omega(P, \rho), \qquad i = 1, 2, \ldots, m \tag{14b}$$

In Ref. 11, the largeness of the ellipsoid $\Omega(P, \rho)$ is measured with respect to a shape reference set X_R. Let $X_R \subset \mathcal{R}^{n+n_c}$ be a prescribed bounded convex set which contains origin. For a set $\mathcal{S} \subset \mathcal{R}^{n+n_c}$, define

$$\alpha_R(\mathcal{S}) := \sup\{\alpha > 0 : \alpha X_R \subset \mathcal{S}\}$$

Obviously, if $\alpha_R(\mathcal{S}) \geq 1$, then $X_R \subset \mathcal{S}$. Two typical types of X_R are the ellipsoid

$$X_R = \{\tilde{x} \in \mathcal{R}^{n+n_c} : \tilde{x}^T R\tilde{x} \leq 1\}, \qquad R > 0 \tag{15}$$

and the polyhedron

$$X_R = \mathrm{cov}\{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_l\} \tag{16}$$

where $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_l$ are the given points in $\mathcal{R}^{n+n_c}$. Then, for any given E_c and reference set X_R, Problem 1 can be solved by the following constrained optimization problem:

$$\max_{P>0, H, \rho>0, \eta>0} \quad \alpha \tag{17}$$

so that

$$\Psi(\boldsymbol{\nu}) < 0, \qquad \forall\boldsymbol{\nu} \in \mathcal{V} \tag{17a}$$

$$|\boldsymbol{h}_i\tilde{x}| \leq 1, \qquad \forall\tilde{x} \in \Omega(P, \rho), \qquad i = 1, 2, \ldots, m \tag{17b}$$

$$\alpha X_R \subset \Omega(P, \rho) \tag{17c}$$

Note that with $Q = (P/\rho)^{-1}$ and $G = HQ$, the matrix inequality in Eq. (13) is equivalent to

$$\bar{\Psi}(\boldsymbol{\nu}) = \bar{\Phi}(\boldsymbol{\nu}) + (\rho/\eta)\tilde{B}_1\tilde{B}_1^T + (\eta/\rho)Q < 0, \qquad \forall\nu \in \mathcal{V} \tag{18}$$

where

$$\bar{\Phi}(\boldsymbol{\nu}) = Q[\tilde{A} - \tilde{B}_2F + \tilde{B}_2M(\boldsymbol{\nu}, F, H)]^T + [\tilde{A} - \tilde{B}_2F + \tilde{B}_2M(\boldsymbol{\nu}, F, H)]Q, \qquad \forall\boldsymbol{\nu} \in \mathcal{V}$$

When it is noted that $M(\boldsymbol{\nu}, F, H)Q = M(\boldsymbol{\nu}, FQ, HQ) = M(\boldsymbol{\nu}, FQ, G)$, then

$$\bar{\Phi}(\boldsymbol{\nu}) = Q(\tilde{A} - \tilde{B}_2F)^T + (\tilde{A} - \tilde{B}_2F)Q + \tilde{B}_2M(\boldsymbol{\nu}, FQ, G) + M^T(\boldsymbol{\nu}, FQ, G)\tilde{B}_2^T \tag{19}$$

Obviously, if ρ/η is fixed, then the matrix inequality in Eq. (18) becomes an LMI.

The constraint in (17b) is equivalent to

$$\rho\boldsymbol{h}_i P^{-1}\boldsymbol{h}_i^T \leq 1 \iff \begin{bmatrix} 1 & \boldsymbol{h}_i Q \\ Q\boldsymbol{h}_i^T & Q \end{bmatrix} \geq 0, \qquad i = 1, 2, \ldots, m$$

$$\iff \begin{bmatrix} 1 & \boldsymbol{g}_i \\ \boldsymbol{g}_i^T & Q \end{bmatrix} \geq 0, \qquad i = 1, 2, \ldots, m$$

where $\boldsymbol{g}_i$ denotes the ith row of G, that is, $\boldsymbol{g}_i = \boldsymbol{h}_i Q$.

If X_R is an ellipsoid given in Eq. (15), then by Schur complement, Eq. (17a) is equivalent to

$$\alpha^{-2}R \geq P/\rho \iff R^{-1} \leq \alpha^{-2}Q$$

In the following, we let $\beta = \alpha^{-2}$. Then, the optimization problem (17) with fixed $\bar{\eta} = \rho/\eta$ can be reduced to the following one with LMI constraints,

$$\min_{Q>0,G} \beta \tag{20}$$

so that

$$\bar{\Psi}(\boldsymbol{\nu}) < 0, \qquad \forall \boldsymbol{\nu} \in \mathcal{V} \tag{20a}$$

$$\begin{bmatrix} 1 & \boldsymbol{g}_i \\ \boldsymbol{g}_i^T & Q \end{bmatrix} \geq 0, \qquad i = 1, 2, \ldots, m \tag{20b}$$

$$R^{-1} \leq \beta Q \tag{20c}$$

To obtain the global maximum of α, we can use a line search by running $\bar{\eta}$ from 0 to ∞. Because ρ can be absorbed into other parameters, we simply set $\rho = 1$.

If X_R is a polyhedron given by Eq. (16), then Eq. (20a) is equivalent to

$$\alpha^2 \tilde{\boldsymbol{x}}_i^T \left(\frac{P}{\rho}\right) \tilde{\boldsymbol{x}}_i \leq 1 \Longleftrightarrow \begin{bmatrix} \alpha^{-2} & \tilde{\boldsymbol{x}}_i^T \\ \tilde{\boldsymbol{x}}_i & Q \end{bmatrix} \geq 0, \qquad i = 1, 2, \ldots, l$$

Then, from the preceding derivation, the optimization problem (17) can be reduced to the following optimization problem with LMI constraints:

$$\min_{Q>0,G} \beta \tag{21}$$

so that

$$\bar{\Psi}(\boldsymbol{\nu}) < 0, \qquad \forall \nu \in \mathcal{V} \tag{21a}$$

$$\begin{bmatrix} 1 & \boldsymbol{g}_i \\ \boldsymbol{g}_i^T & Q \end{bmatrix} \geq 0, \qquad i = 1, 2, \ldots, m \tag{21b}$$

$$\begin{bmatrix} \beta & \tilde{\boldsymbol{x}}_i^T \\ \tilde{\boldsymbol{x}}_i & Q \end{bmatrix} \geq 0, \qquad i = 1, 2, \ldots, l \tag{21c}$$

Invariant Set Enlargement by Antiwindup Compensation

In this section, we will present an iterative design approach to obtain the antiwindup compensation gain such that the invariant set may be as large as possible and, thus, lead to a solution to Problem 2. Clearly, Problem 2 can be formulated in the form of optimization problem (17) with E_c being the extra free parameter.

Note that Eq. (20a) from optimization problem (20) [or Eq. (21a) from problem, (21) respectively] is not linear in E_c, Q, and G simultaneously. We also note that the nonlinear matrix inequality $\Psi(\boldsymbol{\nu}) < 0$ cannot be reduced to an LMI in E_c, P, and H simultaneously. This implies that we cannot compute the antiwindup compensation gain by directly solving an LMI constrained optimization problem. In what follows, we will present an iterative LMI approach to design the antiwindup compensation gain E_c for the set invariance enlargement. Denote

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}, \qquad \tilde{B}_0 = \begin{bmatrix} B_2 \\ 0 \end{bmatrix}, \qquad \tilde{P}_2 = \begin{bmatrix} P_{12} \\ P_{22} \end{bmatrix} \tag{22}$$

where $P_{11} \in \mathcal{R}^{n \times n}$, $P_{22} \in \mathcal{R}^{n_c \times n_c}$, and $P_{12} \in \mathcal{R}^{n \times n_c}$. Then,

$$P\tilde{B}_2 = P\tilde{B}_0 + \tilde{P}_2 E_c$$

and $\Phi(\boldsymbol{\nu})$ can be rewritten as

$$\Phi(\boldsymbol{\nu}) = \tilde{A}^T P + P\tilde{A} + [M(\boldsymbol{\nu}, F, H) - F]^T (P\tilde{B}_0 + \tilde{P}_2 E_c)^T + (P\tilde{B}_0 + \tilde{P}_2 E_c)[M(\boldsymbol{\nu}, F, H) - F]$$

which is linear in P_{11} and E_c. This implies that with fixed P_{22}, P_{12}, and H, we can determine an E_c such that P_{11} is as small as possible, that is, making the region

$$\left\{\boldsymbol{x} \in \mathcal{R}^n : \boldsymbol{x}^T P_{11} \boldsymbol{x} \leq \rho\right\}$$

as large as possible. If X_R is an ellipsoid given in Eq. (15), this problem can be solved by the following LMI optimization problem:

$$\min_{P_1>0,E_c} \beta \tag{23}$$

so that

$$P \leq \beta R \tag{23a}$$

$$\Psi(\boldsymbol{\nu}) < 0, \qquad \forall \boldsymbol{\nu} \in \mathcal{V} \tag{23b}$$

$$\begin{bmatrix} 1 & \boldsymbol{h}_i \\ \boldsymbol{h}_i^T & P \end{bmatrix} \geq 0, \qquad i = 1, 2, \ldots, m \tag{23c}$$

where P and $\tilde{P}_2$ are defined in Eq. (22). If X_R is a polyhedron given in Eq. (16), this problem can be solved by the following optimization problem with LMI constraints:

$$\min_{P_1>0,E_c} \beta \tag{24}$$

so that

$$\beta - \tilde{\boldsymbol{x}}_i^T P \tilde{\boldsymbol{x}}_i \geq 0, \qquad i = 1, 2, \ldots, l \tag{24a}$$

$$\Psi(\boldsymbol{\nu}) < 0, \qquad \forall \boldsymbol{\nu} \in \mathcal{V} \tag{24b}$$

$$\begin{bmatrix} 1 & \boldsymbol{h}_i \\ \boldsymbol{h}_i^T & P \end{bmatrix} \geq 0, \qquad i = 1, 2, \ldots, m \tag{24c}$$

Hence, we can present the following iterative algorithm.

Algorithm 1: The following is an iterative algorithm for determining antiwindup compensation gain.

1) For a given reference set X_R, SOLVE the optimization problem (20) or (21) for $E_c = 0$, denote the solution as β_0, Q_0, and H_0.

2) Set $i = 1$ and E_c with an initial value.

3) SOLVE the optimization problem (20) or (21) for β, Q, and G. Denote the solution as β_i, Q, and G, respectively.

4) Let $P = Q^{-1}$ and $H = GQ^{-1}$.

5) IF $\beta_i - \beta_{i-1} < \delta$, a predetermined tolerance, GOTO step 7, ELSE GOTO step 6.

6) SOLVE the optimization problem (23) or (24) for P_{11} and E_c with the fixed P_{12}, P_{22}, and H. Set the solution as E_c and $i = i + 1$, then GOTO step 3.

7) IF $\beta_i \leq \beta_0$, then, $\alpha_{\text{opt}} = (\beta_i)^{-1/2}$ and E_c is a feasible solution and STOP, ELSE set E_c with another initial value and GOTO step 2.

Remark 1: It is easy to see that β_i is decreasing and bounded. Hence, the algorithm always converges. As is usual in nonlinear optimization problems, the optimization result depends on the given initial condition of E_c. If we set 0 as the initial values of E_c in step 2, we can always obtain a better solution than the system without antiwindup compensation, although it may not result in the best α_{opt}. In general, we can select several typical initial values of E_c and choose the biggest α_{opt} and the corresponding E_c as the optimal solution.

Remark 2: In practical control design, it is always desirable to design an antiwindup compensation gain E_c with limited value. This implies that with $E_c = [e_{ij}]_{n_c \times m}$, the compensation gain may be constrained element by element in the following form

$$\varphi_{ij} \leq e_{ij} \leq \psi_{ij}, \qquad i = 1, 2, \ldots, n_c, \qquad j = 1, 2, \ldots, m \tag{25}$$

Note that the present technique uses E_c as a free design variable in the LMI optimization problems (23) and (24). These constraints are linear and, hence, can also be incorporated naturally in the optimization process. These extra constraints can also prevent the numerical stiffness.

$\mathcal{L}_2$-Gain Analysis

In this section, we will propose a solution to Problem 3. We will do this in two steps. First, we assume that an antiwindup compensation gain E_c has been designed. The problem of determining the largest invariant set in which the closed-loop system has a prespecified regional $\mathcal{L}_2$ gain $\gamma > \gamma_0$ is formulated into and solved as an optimization problem. Then, the same optimization problem is solved with E_c as an extra free parameter. This, thus, leads to a solution to Problem 3.

Invariant Set with a Given $\mathcal{L}_2$ Gain

We first establish the following result.

Theorem 3: Given system (1–3), suppose that the control law is designed such that the closed-loop system (8–10), without input saturation, is asymptotically stable and has an $\mathcal{L}_2$ gain less than or equal to γ. Then, if there exist matrices $H \in \mathcal{R}^{m\times(n+n_c)}$ and $P > 0$ and $\eta > 0$ that satisfy matrix inequalities (13) and

$$\Theta(\boldsymbol{\nu}) = \begin{bmatrix} \Phi(\boldsymbol{\nu}) + \tilde{C}_1^T\tilde{C}_1 & P\tilde{B}_1 + \tilde{C}_1^T D_1 \\ \tilde{B}_1^T P + D_1^T\tilde{C}_1 & -\gamma^2 I + D_1^T D_1 \end{bmatrix} < 0, \qquad \forall\boldsymbol{\nu} \in \mathcal{V} \tag{26}$$

where

$$\Phi(\boldsymbol{\nu}) = [\tilde{A} - \tilde{B}_2 F + \tilde{B}_2 M(\boldsymbol{\nu}, F, H)]^T P + P[\tilde{A} - \tilde{B}_2 F + \tilde{B}_2 M(\boldsymbol{\nu}, F, H)] \tag{27}$$

and $\Omega(P, \rho) \subset \mathcal{L}(H)$. Then:

1) In the absence of disturbance, the closed-loop system (8–10) with input saturation is locally asymptotically stable with $\Omega(P, \rho)$ contained in its domain of attraction.

2) The closed-loop system has a regional $\mathcal{L}_2$ gain less than or equal to γ in $\Omega(P, \rho)$.

Proof: Obviously, if Eq. (26) holds for all $\boldsymbol{\nu} \in \mathcal{V}$, then $\Phi(\boldsymbol{\nu}) < 0$ for all $\boldsymbol{\nu} \in \mathcal{V}$. By Theorem 2.2, matrix inequalities (13) establishes an invariant ellipsoid that overbounds the reachable set under the disturbance $\boldsymbol{w}$.

Next, select a Lyapunov function $V(\tilde{\boldsymbol{x}}) = \tilde{\boldsymbol{x}}^T P\tilde{\boldsymbol{x}}$. Following the procedure of the proof of Theorem 1 in Ref. 11, we can show that for each $\tilde{\boldsymbol{x}} \in \Omega(P, \rho) \subset \mathcal{L}(H)$, there exists a $\boldsymbol{\nu} \in \mathcal{V}$ such that

$$2\tilde{\boldsymbol{x}}^T P\{\tilde{A}\tilde{\boldsymbol{x}} + \tilde{B}_2[\sigma(\boldsymbol{u}) - \boldsymbol{u}]\} \le \tilde{\boldsymbol{x}}^T\Phi(\boldsymbol{\nu})\tilde{\boldsymbol{x}}$$

Because for each $\tilde{\boldsymbol{x}} \in \Omega(P, \rho)$,

$$\dot{V} = 2\tilde{\boldsymbol{x}}^T P\{\tilde{A}\tilde{\boldsymbol{x}} + \tilde{B}_2[\sigma(\boldsymbol{u}) - \boldsymbol{u}] + \tilde{B}_1\boldsymbol{w}\} \le \tilde{\boldsymbol{x}}^T\Phi(\boldsymbol{\nu})\tilde{\boldsymbol{x}} + 2\tilde{\boldsymbol{x}}^T P\tilde{B}_1\boldsymbol{w}$$

we have

$$J = -[\gamma^2\|\boldsymbol{w}(t)\|^2 - \|z(t)\|^2] + \dot{V}(\boldsymbol{x}) \le \tilde{\boldsymbol{x}}^T\tilde{C}_1^T\tilde{C}_1\tilde{\boldsymbol{x}} + \tilde{\boldsymbol{x}}^T\tilde{C}_1 D_{11}\boldsymbol{w} + \boldsymbol{w}^T D_{11}^T\tilde{C}_1\tilde{\boldsymbol{x}} + \boldsymbol{w}^T\left(D_{11}^T D_{11} - \gamma^2 I\right)\boldsymbol{w} + \tilde{\boldsymbol{x}}^T\Phi(\boldsymbol{\nu})\tilde{\boldsymbol{x}} + 2\tilde{\boldsymbol{x}}^T P\tilde{B}_1\boldsymbol{w} = \hat{\boldsymbol{x}}^T\Theta(\boldsymbol{\nu})\hat{\boldsymbol{x}}$$

for each $\tilde{\boldsymbol{x}} \in \Omega(P, \rho)$, where

$$\hat{\boldsymbol{x}}(t) = [\tilde{\boldsymbol{x}}^T(t) \quad \boldsymbol{w}^T(t)]^T$$

Hence, if for each $\boldsymbol{\nu} \in \mathcal{V}$, $\Theta(\boldsymbol{\nu}) < 0$, then we have $J < 0$. This implies that for each $\tilde{\boldsymbol{x}} \in \Omega(P, \rho)$ and $\boldsymbol{w}(t) \in \mathcal{L}_2[0, \infty)$,

$$\dot{V}(x) \le \gamma^2\|\boldsymbol{w}(t)\|^2 - \|z(t)\|^2$$

that is, the closed-loop system has a regional $\mathcal{L}_2$ gain less than or equal to γ. □

Based on Theorem 3, we can present the following optimization problem to determine the possible smallest $\mathcal{L}_2$ gain of the closed-loop system in some invariant set $\Omega(P, \rho)$. We will denote such a gain γ_{opt}:

$$\min_{P>0,H,\rho} \bar{\gamma} = \gamma^2 \tag{28}$$

so that,

$$\Psi(\boldsymbol{\nu}) < 0, \qquad \forall\boldsymbol{\nu} \in \mathcal{V} \tag{28a}$$

$$\Theta(\boldsymbol{\nu}) < 0, \qquad \forall\boldsymbol{\nu} \in \mathcal{V} \tag{28b}$$

$$|h_i\tilde{x}| \le 1, \qquad \forall\tilde{x} \in \Omega(P, \rho), \qquad i = 1, 2, \ldots, m \tag{28c}$$

With the solution P and $\bar{\gamma}$, we know that the closed-loop system is locally asymptotically stable and has a reginal $\mathcal{L}_2$ gain less than or equal to $\gamma_{\text{opt}} = \sqrt{\bar{\gamma}}$ in $\Omega(P, \rho)$.

Obviously, even with a given compensation gain E_c, Eq. (26) is not an linear matrix inequality in P, H, and γ. Fortunately, with $Q = (P/\rho)^{-1}$ and $G = HQ$, it can be reduced to the following LMI:

$$\bar{\Theta}(\nu) = \begin{bmatrix} \bar{\Phi}(\boldsymbol{\nu}) & \rho\tilde{B}_1 & Q\tilde{C}_1^T \\ \rho\tilde{B}_1^T & -\rho\gamma^2 I & \rho D_1^T \\ \tilde{C}_1 Q & \rho D_1 & -\rho I \end{bmatrix} < 0, \qquad \forall\nu \in \mathcal{V} \tag{29}$$

It is easy to find that the optimization problem (28) for some fixed $\bar{\eta} = \rho/\eta$ can be reduced to the following optimization problem:

$$\min_{P>0,G,\rho} \gamma^2 \tag{30}$$

so that,

$$\bar{\Psi}(\boldsymbol{\nu}) < 0, \qquad \forall\boldsymbol{\nu} \in \mathcal{V} \tag{30a}$$

$$\bar{\Theta}(\boldsymbol{\nu}) < 0, \qquad \forall\boldsymbol{\nu} \in \mathcal{V} \tag{30b}$$

$$\begin{bmatrix} 1 & \boldsymbol{g}_i \\ \boldsymbol{g}_i^T & Q \end{bmatrix} \ge 0, \qquad i = 1, 2, \ldots, m \tag{30c}$$

With the solution of the preceding optimization problem, we know that the saturated closed-loop system is locally asymptotically stable and has a regional $\mathcal{L}_2$ gain less than or equal to γ in the region $\Omega(Q^{-1}, 1)$. The global minimum of γ, γ_{opt}, will be obtained by running $\bar{\eta}$ from 0 to ∞.

Theorem 3 gives a condition for an ellipsoid to be inside the domain of attraction of the saturated system with the given $\mathcal{L}_2$ gain less than or equal to $\gamma \ge \gamma_{\text{opt}}$. With the preceding reference sets, we can choose from all of the $\Omega(P, \rho)$ that satisfy the condition, such that the quantity $\alpha_R[\Omega(P, \rho)]$ is maximized. This problem can be formulated as

$$\max_{P>0,\rho,H} \alpha \tag{31}$$

so that

$$\alpha X_R \subset \Omega(P, \rho) \tag{31a}$$

$$\Psi(\boldsymbol{\nu}) < 0, \qquad \forall\boldsymbol{\nu} \in \mathcal{V} \tag{31b}$$

$$\Theta(\boldsymbol{\nu}) < 0, \qquad \forall\boldsymbol{\nu} \in \mathcal{V} \tag{31c}$$

$$|\boldsymbol{h}_i\tilde{\boldsymbol{x}}| \le 1, \qquad \forall\tilde{\boldsymbol{x}} \in \Omega(P, \rho), \qquad i = 1, 2, \ldots, m \tag{31d}$$

With the given compensation gain E_c, the preceding optimization constraints can be reduced to some LMIs. If X_R is an ellipsoid given in Eq. (15), then the optimization problem (31) with fixed $\bar{\eta}$ can be reduced to the following one with LMI constraints:

$$\min_{Q>0,G,\rho>0} \beta \tag{32}$$

so that

$$R^{-1} \le \beta Q \tag{32a}$$

$$\bar{\Psi}(\boldsymbol{\nu}) < 0, \qquad \forall\boldsymbol{\nu} \in \mathcal{V} \tag{32b}$$

$$\bar{\Theta}(\boldsymbol{\nu}) < 0, \qquad \forall\boldsymbol{\nu} \in \mathcal{V} \tag{32c}$$

$$\begin{bmatrix} 1 & \boldsymbol{g}_i \\ \boldsymbol{g}_i^T & Q \end{bmatrix} \ge 0, \qquad i = 1, 2, \ldots, m \tag{32d}$$

Hence, with the solution of LMI optimization problem (32), $\Omega(Q^{-1}, 1)$ is an estimation of the invariant set with guaranteed $\mathcal{L}_2$ gain $\gamma \ge \gamma_{\text{opt}}$ of the saturated closed-loop system with given compensation gain E_c.

If X_R is a polyhedron given by Eq. (16), then, from the preceding derivation, the optimization problem (31) can be reduced to the following problem with LMI constraints:

$$\min_{Q>0,G,\rho} \beta \tag{33}$$

so that

$$\begin{bmatrix} \beta & \tilde{x}_i^T \\ \tilde{x}_i & Q \end{bmatrix} \geq 0, \qquad i = 1, 2, \ldots, l \tag{33a}$$

$$\bar{\Psi}(\nu) < 0, \qquad \forall \nu \in \mathcal{V} \tag{33b}$$

$$\bar{\Theta}(\nu) < 0, \qquad \forall \nu \in \mathcal{V} \tag{33c}$$

$$\begin{bmatrix} 1 & g_i \\ g_i^T & Q \end{bmatrix} \geq 0, \qquad i = 1, 2, \ldots, m \tag{33d}$$

With the solution of the preceding LMI optimization problem, the invariant set with guaranteed $\mathcal{L}_2$ gain γ is $\Omega(Q^{-1}, 1)$.

Antiwindup Compensation Gain Design

In this section, we will present an iterative design approach to obtain the antiwindup compensation gain E_c such that, for any given $\gamma \geq \gamma_{\text{opt}}$, the closed-loop system has a regional $\mathcal{L}_2$ gain less or equal to γ in some $\Omega(P, \rho) \supset \alpha X_R$ with α maximized.

Note that conditions in optimization problem (32b) and (32c) [or problem (33), respectively] are not linear in ρ (or ρ^{-1}, respectively), E_c, Q, and G simultaneously. We also note that the nonlinear matrix inequality (29) cannot be reduced to an LMI in ρ (or ρ^{-1}), E_c, P, and H simultaneously. This implies that we cannot obtain the antiwindup compensation gain by directly solving an LMI optimization problem. In what follows, we will present an iterative LMI approach to design the antiwindup compensation gain E_c. Similar to the last section, we may first fix E_c and γ and apply the LMI optimization problem to obtain a less conservative estimation of the invariant set and the corresponding ellipsoid. Then, with the fixed ρ, P_{22}, P_{12}, and H, we can determine an E_c such that P_{11} is as small as possible, that is, making the region $\{x \in \mathcal{R}^n : x^T P_{11} x \leq \rho\}$ as large as possible. If X_R is an ellipsoid given in Eq. (15), this problem can be solved by the following LMI optimization problem:

$$\min_{P_{11} > 0, E_c} \beta \tag{34}$$

so that

$$\rho^{-1} P \leq \beta R \tag{34a}$$

$$\Psi(\nu) < 0, \qquad \forall \nu \in \mathcal{V} \tag{34b}$$

$$\Theta(\nu) < 0, \qquad \forall \nu \in \mathcal{V} \tag{34c}$$

$$\begin{bmatrix} \rho^{-1} & h_i \\ h_i^T & P \end{bmatrix} \geq 0, \qquad i = 1, 2, \ldots, m \tag{34d}$$

where P and $\tilde{P}_2$ are defined in Eq. (22). If X_R is a polyhedron given in Eq. (16), this problem can be solved by the following LMI optimization problem:

$$\min_{P_{11} > 0, E_c} \beta \tag{35}$$

so that

$$\rho\beta - \tilde{x}_i^T P \tilde{x}_i \geq 0, \qquad i = 1, 2, \ldots, l \tag{35a}$$

$$\Psi(\nu) < 0, \qquad \forall \nu \in \mathcal{V} \tag{35b}$$

$$\Theta(\nu) < 0, \qquad \forall \nu \in \mathcal{V} \tag{35c}$$

$$\begin{bmatrix} \rho^{-1} & h_i \\ h_i^T & P \end{bmatrix} \geq 0, \qquad i = 1, 2, \ldots, m \tag{35d}$$

Hence, we can present the following iterative algorithm.

Algorithm 2: The iterative algorithm for determining antiwindup compensation gain consists of the following:

1) For a given reference set X_R, SOLVE the optimization problem (32) or (33) for $E_c = 0$ and denote the solution as β_0, ρ_0, Q_0, and H_0.

2) Set $i = 1$ and E_c with an initial value.

3) SOLVE the optimization problem (32) or (33) for β, ρ, Q, and G. Denote the solution as β_i, ρ, Q, and G, respectively.

4) Let $P = \rho Q^{-1}$ and $H = GQ^{-1}$.

5) IF $\beta_i - \beta_{i-1} < \delta$, a predetermined tolerance, GOTO step 7, ELSE GOTO step 6.

6) SOLVE the optimization problem (34) or (35) for P_{11} and E_c with the fixed ρ, P_{12}, P_{22}, and H. Set the solution as E_c and $i = i + 1$, then GOTO step 3.

7) IF $\beta_i \leq \beta_0$, then, $\alpha_{\text{opt}} = (\beta_i)^{-1/2}$ and E_c is a feasible solution and STOP, ELSE set E_c with another initial value and GOTO step 2.

Remark 3: In this iterative algorithm, ρ has to be a free parameter because it cannot be absorbed into other parameters.

Numerical Examples

Example 1: Consider the following benchmark example,[16]

$$\dot{x}_1 = -0.1x_1 + 0.5\,\text{sat}(u_1) + 0.4\,\text{sat}(u_2)$$

$$\dot{x}_2 = -0.1x_2 + 0.4\,\text{sat}(u_1) + 0.3\,\text{sat}(u_2)$$

where u_1 and u_2 range between $[-3, 3]$ and $[-10, 10]$, respectively. At time $t = 0$, the outputs x_1 and x_2 are subject to pulse set-point changes of duration 5 s and magnitudes 0.6 and 0.4, respectively. The following PI controller was considered in Ref. 16:

$$\dot{x}_{c1} = y_{\text{sp1}} - x_1 + e_{11}[\text{sat}(u_1) - u_1] + e_{12}[\text{sat}(u_2) - u_2]$$

$$\dot{x}_{c2} = y_{\text{sp2}} - x_2 + e_{21}[\text{sat}(u_1) - u_1] + e_{22}[\text{sat}(u_2) - u_2]$$

$$u_1 = 10(y_{\text{sp1}} - x_1) + x_{c1}$$

$$u_2 = -10(y_{\text{sp2}} - x_2) - x_{c2}$$

where y_{sp1} and y_{sp2} refer to the set points for the outputs, which places the closed-loop poles at $(-1, -1, -0.1, -0.1)$ in the absence of saturation. In Ref. 19 a new design approach is presented, and the comparison results are given by simulation.

To apply our result, we set

$$A = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.1 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0.5 & 0.4 \\ 0.4 & 0.3 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} 1.5 & 4 \\ 1.2 & 3 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad A_c = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad B_c = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$C_c = \left(\begin{bmatrix} 3 & 0 \\ 0 & 10 \end{bmatrix}\right)^{-1}\begin{bmatrix} 1.0 & 0 \\ 0 & -1.0 \end{bmatrix} = \begin{bmatrix} 0.3333 & 0 \\ 0 & -0.1 \end{bmatrix}$$

$$D_c = \left(\begin{bmatrix} 3 & 0 \\ 0 & 10 \end{bmatrix}\right)^{-1}\begin{bmatrix} -10 & 0 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} -3.333 & 0 \\ 0 & 1 \end{bmatrix}$$

To determine the disturbance rejection and $\mathcal{L}_2$ performance, we let

$$B_1 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \qquad C_1 = [-1 \quad -1], \qquad D_1 = 0$$

Let

$$X_R = \begin{bmatrix} x_0 \\ x_c(0) \end{bmatrix}$$

Based on the possible setpoint changes, we set

$$x_0 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}, \qquad x_c(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Let

$$E_c = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Applying Theorem 2, we obtain $\alpha = 0.6124$ at $\bar{\eta} = 5.1$ and the maximal invariant ellipsoid $\Omega(P^0, 1)$ with

$$P^0 = \begin{bmatrix} 61.1518 & -68.2916 & -6.1152 & 6.8292 \\ -68.2916 & 83.9480 & 6.8292 & -8.3948 \\ -6.1152 & 6.8292 & 0.6115 & -0.6829 \\ 6.8292 & -8.3948 & -0.6829 & 0.8395 \end{bmatrix}$$

Table 1 Computing results with given $\mathcal{L}_2$ gain

γ	α_0	α	E_c	P	ρ
0.197	0.1157	0.2059	$\begin{bmatrix} 41.29 & -13.65 \\ 39.87 & -13.43 \end{bmatrix}$	$\begin{bmatrix} 249.4 & -258.5 & -25.33 & 25.46 \\ -258.5 & 271.8 & 26.34 & -26.69 \\ -25.33 & 26.34 & 2.660 & -2.507 \\ 25.46 & -26.69 & -2.507 & 2.708 \end{bmatrix}$	0.3907
0.2	0.1778	0.2476	$\begin{bmatrix} 22.70 & -16.59 \\ 21.93 & -18.16 \end{bmatrix}$	$\begin{bmatrix} 175.61 & -184.4 & -17.91 & 18.09 \\ -184.4 & 197.6 & 18.90 & -19.30 \\ -17.91 & 18.90 & 1.914 & -1.767 \\ 18.09 & -19.30 & -1.767 & 1.972 \end{bmatrix}$	0.3869
0.5	0.5649	0.7735	$\begin{bmatrix} 17.85 & -26.22 \\ 17.10 & -25.35 \end{bmatrix}$	$\begin{bmatrix} 146.2 & -161.3 & -15.45 & 15.31 \\ -161.3 & 195.8 & 17.15 & -18.55 \\ -15.45 & 17.15 & 1.686 & -1.575 \\ 15.31 & -18.55 & -1.575 & 1.811 \end{bmatrix}$	3.917
1	0.6104	1.2267	$\begin{bmatrix} 39.92 & -41.01 \\ 36.51 & -40.02 \end{bmatrix}$	$\begin{bmatrix} 781.2 & -838.8 & -78.73 & 83.27 \\ -838.8 & 987.6 & 84.66 & -97.98 \\ -78.73 & 84.66 & 7.709 & -8.351 \\ 83.27 & -97.98 & -8.351 & 9.774 \end{bmatrix}$	55.10
2	0.6129	2.3482	$\begin{bmatrix} 43.97 & -38.45 \\ 39.10 & -38.00 \end{bmatrix}$	$\begin{bmatrix} 3617 & -3939 & -362.1 & 393.5 \\ -3939 & 4665 & 394.4 & -465.9 \\ -362.1 & 394.4 & 36.3 & -39.3 \\ 393.5 & -465.9 & -39.3 & 46.6 \end{bmatrix}$	870.4

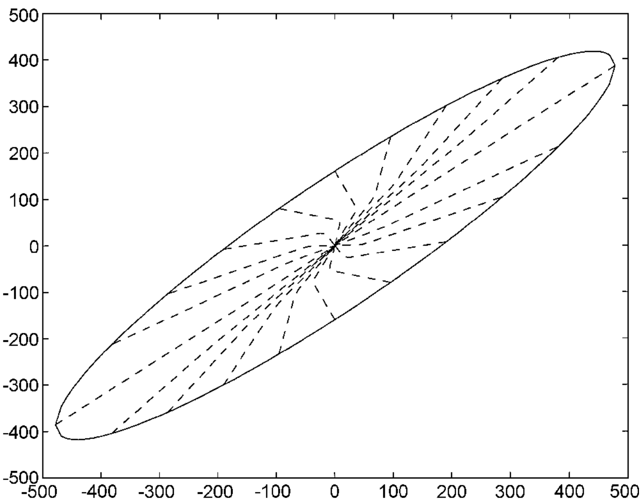

Fig. 1 Maximal invariant ellipsoid and the state trajectories of Example 1.

This ellipsoid is very small and does not include the initial point just given. This means that the closed-loop system trajectory may be unbounded with the preceding set-point change with disturbance. Applying Algorithm 1 with initial

$$E_c = \begin{bmatrix} 10 & -10 \\ 10 & -10 \end{bmatrix}$$

we obtain following solution:

$$\alpha = 798.8314, \qquad E_c = \begin{bmatrix} 19.1177 & -91.3001 \\ 18.8603 & -99.4319 \end{bmatrix}$$

$$P = 10^{-5} \times \begin{bmatrix} 2.1700 & -2.1535 & -0.2194 & 0.2144 \\ -2.1535 & 2.5696 & 0.2183 & -0.2548 \\ -0.2194 & 0.2183 & 0.0231 & -0.0226 \\ 0.2144 & -0.2548 & -0.0226 & 0.0263 \end{bmatrix}$$

when $\bar{\eta} = 5.6$. This ellipsoid $\Omega(P, 1)$ with $\boldsymbol{x}_c = 0$ is shown in Fig. 1. The dashed curves are the state trajectories with the state initial conditions on the boundary of this ellipsoid, and the controller initial state $\boldsymbol{x}_c(0) = 0$. All trajectories converge to the origin.

It is easy to check that in the absence of saturation, the $\mathcal{L}_2$ gain of the closed-loop system with the given controller is $\gamma_0 = 0.1825$. In the presence of saturation, the optimization algorithm (30), we obtain the optimal solution

$$\gamma_{\text{opt}} = 0.197, \qquad \rho = 0.3889$$

$$P = 10^7 \times \begin{bmatrix} 2.4927 & -2.4927 & -0.2493 & 0.2493 \\ -2.4927 & 2.4927 & 0.2493 & -0.2493 \\ -0.2493 & 0.2493 & 0.0249 & -0.0249 \\ 0.2493 & -0.2493 & -0.0249 & 0.0249 \end{bmatrix}$$

when $\eta = 0.0194$. Note that $\gamma_{\text{opt}} > \gamma_0$, and this ellipsoid is very small. This means that the input saturation leads to very obvious deterioration in the performance of the control system. In the following, we will use the result of the preceding section to improve the system performance.

Using the preceding data as the initial condition to apply the iterative algorithm of the last section, and setting the gain constraint $\psi_{ij} = -\varphi_{ij} = 100$ for some given γ, we obtain the computing results showing in Table 1.

From Table 1, even we relax the required $\mathcal{L}_2$ gain $\gamma = 2$, which is more than 10 times the $\mathcal{L}_2$ gain of the original unsaturated closed-loop system, the minimum α_0 corresponding to $E_c = 0$ is still smaller than 1, which means that the domain of attraction with the given $\mathcal{L}_2$ gain can not include the given initial point x_0. Fortunately, under the preceding iterative algorithm, we can enlarge the domain of the attraction for any required H_∞ performance with the help of antiwindup compensation. With $\gamma = 1$, we have $\alpha > 1$, which means that the corresponding invariant set $\Omega(P, \rho)$ includes the preceding initial point, and, hence, the stability of the preceding set-point changes are guaranteed, and the disturbance rejection performance may be satisfied.

Example 2: The following state equations describe the longitudinal dynamics of the F-8 fighter aircraft,[26]

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -0.8 & -0.006 & -12 & 0 \\ 0 & -0.014 & -16.64 & -32.2 \\ 1 & -0.0001 & -1.5 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} -19 & -3 \\ -0.66 & -0.5 \\ -0.16 & -0.5 \\ 0 & 0 \end{bmatrix} \boldsymbol{u} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \boldsymbol{w}$$

$$\boldsymbol{y} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \boldsymbol{x}$$

$$\boldsymbol{z} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \boldsymbol{w}$$

The input, output, and state variables are given as follows:

$$\boldsymbol{u} = \begin{cases} \delta_e & \text{elevator angle (degrees), saturation limit at 25 deg} \\ \delta_f & \text{flaperon angle (degrees), saturation limit at 25 deg} \end{cases}$$

$$\boldsymbol{y} = \begin{cases} \vartheta & \text{pitch angle (radians)} \\ \gamma & \text{flight path angle (radians)} \end{cases}$$

$$\boldsymbol{x} = \begin{cases} q & \text{pitch rate (radians per second)} \\ v & \text{forward velocity (feet per second)} \\ \alpha & \text{angle of attack (radians)} \\ \vartheta & \text{pitch angle (radians)} \end{cases}$$

and $\boldsymbol{w}$ represents the disturbance. This system has an unstable pole at 0.14. The control system output $\boldsymbol{y}$ is required to track reference

$$\boldsymbol{r} = [10 \quad 10]^T$$

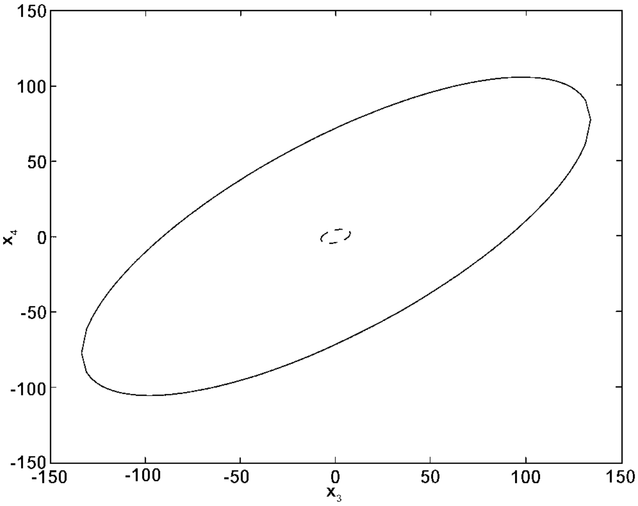

Fig. 2 Invariant ellipsoids for the F8 system with saturation: ——, with antiwindup and – – –, without antiwindup.

with a steady-state tracking error less than 2%. An antiwindup control scheme was designed in Ref. 26 using an error governor compensation that scales down the control input in the presence of saturation. However, the proposed scheme is only applicable to open-loop stable systems, and the controller is also constrained to be stable. Here, we use the observer-based antiwindup scheme to design an antiwindup compensation gain to cause the system output to track the reference as large as possible in the presence of saturation.

We first use the linear quadratic Gaussion methodology to design the observer-based compensator in the absence of saturation, which is computed as follows:

$$K(s) = G(sI - A - BG - HC)^{-1}H$$

where

$$H = \begin{bmatrix} 0.3673 & -39.7859 & 0.1344 & 0.9961 \\ 0.4944 & -18.7975 & -0.1059 & 0.8617 \end{bmatrix}^T$$

$$G = \begin{bmatrix} 10.2575 & -7.2739 & 14.4702 & 48.0214 \\ 1.8413 & -6.8236 & 24.3056 & 40.4592 \end{bmatrix}$$

We then use Theorem 2 to study the disturbance rejection performance in the presence of saturation. We set

$$\mathcal{X}_R = \text{cov}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

By solving optimization problem (21), we obtain $\alpha_0 = 4.1636$ with $\bar{\eta} = 49.5$. The ellipsoid with $\boldsymbol{x}_1 = 0$ and $\boldsymbol{x}_2 = 0$ is shown in Fig. 2 by a dashed curve. Because $\alpha_0 < 10$, the stability cannot be guaranteed when we track the reference

$$r = [10 \quad 10]^T$$

in the presence of saturation. Actually, it is easy to check that, under this controller, the outputs cannot track the reference

$$r = [10 \quad 10]^T$$

See Figs. 3b and 3d.

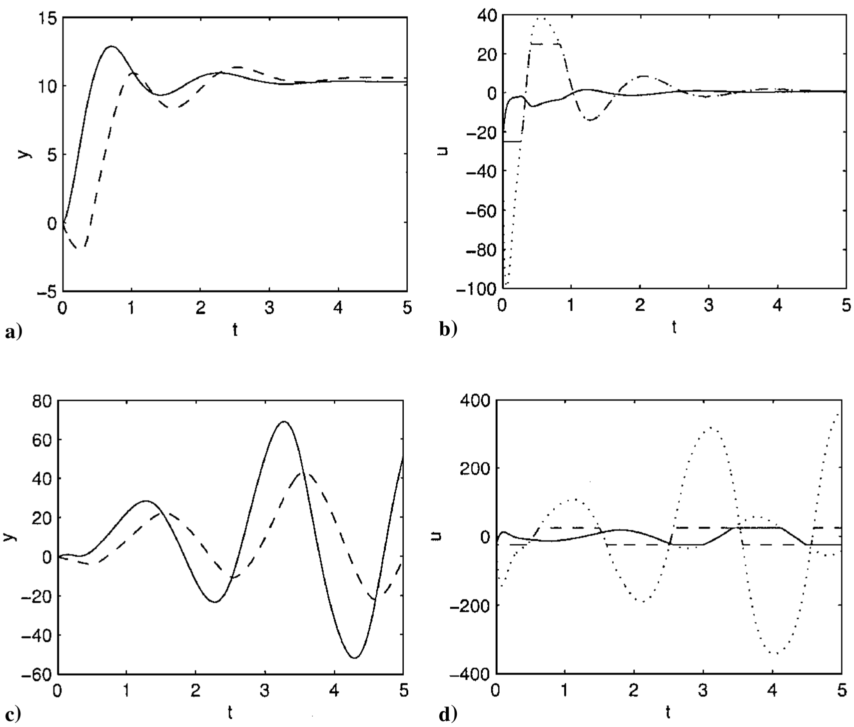

Fig. 3 Input and output responses for the F8 system with saturation: a) and c) with antiwindup compensation and b) and d) without antiwindup compensation.

We next use the proposed iterative algorithm to design an antiwindup gain to compensate the actuator saturation. To apply the iterative algorithm, we set the initial values of E_c be

$$E_c^0 = \begin{bmatrix} 10 & 10 & 10 & 10 \\ 10 & 10 & 10 & 10 \end{bmatrix}^T$$

We obtain

$$\alpha = 72.0719, \qquad E_c = \begin{bmatrix} -3.8075 & -3.6665 \\ -3.9756 & -3.9928 \\ -0.6943 & -0.0973 \\ -0.5626 & -0.0367 \end{bmatrix}$$

The ellipsoid with $\boldsymbol{x}_1 = 0$ and $\boldsymbol{x}_2 = 0$ is shown in Fig. 2 by the solid curve. It is obvious that the ellipsoid is significantly enlarged with α increased almost 15 times. Thus, with this antiwindup compensation, the output is able to track the reference

$$\boldsymbol{r} = [10 \quad 10]^T$$

See Figs. 3a and 3c.

We can check that the $\mathcal{H}_\infty$ performance of the system is also improved. In the absence of antiwindup compensation, by solving optimization problem (33), we find that the largest α is 2.9475 with $\gamma = 1.1$. With the aforementioned antiwindup compensation, α can be enlarged to 30.3977 when $\gamma = 1.1$. This implies that with the same $\mathcal{L}_2$-gain constraint, the antiwindup compensation can significantly increase the size of the invariant set with guaranteed performance.

Conclusions

In this paper, we have proposed to use the antiwindup design technique to improve the invariant set with given H_∞ performance index for a predesigned closed-loop system under actuator saturation. Stability and the invariant set of the closed-loop system with actuator saturation and an antiwindup compensator are analyzed. An estimation method for the invariant set is introduced using LMI techniques. An iterative algorithm is presented to design the antiwindup compensation gain such that the invariant set of the closed-loop system is enlarged in the presence of saturation. The $\mathcal{L}_2$ gain of the closed-loop system with antiwindup compensation is analyzed, and the iterative algorithm is extended to enlarge the invariant set with the given $\mathcal{L}_2$ gain by the antiwindup compensation. The application of the proposed analysis and design method to an F8 fighter aircraft shows that the antiwindup design indeed leads to a significant improvement of the closed-loop system stability to track large command inputs under actuator saturation and external disturbance.

Acknowledgment

This work was supported in part by NASA Langley Research Center.

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